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NATURAL VIBRATION OF A BEAM–WATER INTERACTION SYSTEM

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The dynamical behaviour of a flexible beam–water interaction system is examined. The coupled system is subject to an undisturbed boundary condition at infinity in the water domain and a zero surface wave or linear surface disturbance condition on the free surface. The governing equations describing the behaviour of the system are analyzed by using the separation of variables method and their solutions presented. The eigenvalue equation of the natural vibration of the beam–water system is derived and exact solutions for each combination of boundary conditions are obtained. Calculations show that for the undisturbed condition at infinity in the water domain, the natural frequencies of the coupled dynamic system are lower than those of the flexible dry beam, indicating that the influence of water on the beam has the effect of an additional mass. It is further shown that the free surface wave disturbance plays a more important role in the determination of vibration characteristics in the lower frequency region of the coupled system and that fluid compressibility is more influential at higher frequencies. The orthogonality relation of the natural vibration forms of the coupled fluid–structure interaction system are derived and the case of this coupled system subject to the radiation condition at infinity proposed by Sommerfeld [1] is discussed.

1. INTRODUCTION

In offshore and irrigation engineering, a structure such as an offshore platform, a dam or a tower surrounded by water is usually simplified in analyses as a beam or a column structure when determining its static or dynamic responses. Therefore, the flexible beam–water interaction problem is of significance in these branches of engineering. Westergaard [2] first investigated the hydrodynamic pressure on a rigid dam during earthquakes, although the effect of surface waves is ignored in this classical study.

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Westergaard's findings revealed that the magnitude of the hydrodynamic pressure depends on the excitation frequency. Chopra [3] derived an analytical solution of the hydrodynamic pressure on a vertical rigid dam and showed that Westergaard's solutions are valid only if the excitation frequency is less than the fundamental frequency of the reservoir. He extended the theory to investigate the hydrodynamic pressure resulting from horizontal and vertical ground motions including the influence of free surface waves. Chopra concluded that the associated contribution of free surface wave disturbance is small. Chwang [4] presented an exact solution of the hydrodynamic pressure on a rigid dam with an inclined upstream face of constant slope but neglected the compressibility of the fluid in the reservoir. Liu [5] extended Chwang's work to obtain an exact solution for a rigid sloping embankment damming a triangular shaped reservoir. Xing and Price [6] re-examined the influence of free surface waves on the hydrodynamic pressures experienced by dams during an earthquake tremor. It was assumed that this excitation caused a sinusoidal horizontal vibration in the dam and a sinusoidal vertical vibration over a prescribed floor region within the reservoir. It was shown that the two free surface boundary conditions produce only small differences in the values of the natural frequencies.

In the previously described investigations, the elasticity or flexibility of the structure was not considered. Inclusion of this effect complicates the problem significantly. Therefore, to simplify this dynamic problem, the effects of free surface waves or fluid compressibility or both are often neglected (see, for example, references [7, 8]). Assuming an undisturbed condition at infinity in the water domain, the coupled vibrations between a flexible column structure and water including the effects of surface wave disturbance and fluid compressibility were studied by Goto and Toki [9], Liaw and Chopra [10] and Zhu, Weng and Wu [11]. These investigations showed that the influence of free surface waves is of greater importance to the dynamical behaviour of a long-thin beam–water system, whereas the effect of fluid compressibility is the more dominant influence for a short-thick beam–water system.

In a hydrodynamic analysis, the radiation condition at infinity in the water domain plays an important role in determining the behaviour characteristics of the fluid. In developing exact solutions of the hydrodynamic loadings on rigid dams excited by horizontal and vertical vibrations, Xing and Price [6] concluded that at infinity in the water domain the nth component of the dynamic pressure response in the horizontal excitation case satisfies the undisturbed condition if the nth natural frequency of the reservoir is higher than the excitation frequency or the radiation condition if it is less than the excitation frequency. Therefore, if the frequency of excitation is higher than the fundamental freqency of the reservoir, the components of the dynamic pressure response are a combination of two types: namely, one satisfying the undisturbed condition and the other satisfying the radiation condition, both at infinity. However, these conclusions are associated with rigid structures, since distortions of the structures were excluded. It is interesting to note that Chopra [3] concluded that Westergaard's classic solutions are valid only if the frequency of excitation is less than the fundamental frequency of the reservoir, which—in the context of the previous discussion—implies that solutions relate to the undisturbed or zero disturbance condition at infinity in the fluid domain.

In both two- and three-dimensional time domain analyses of fluid–structure interactions, Tsai and Lee [12, 13], Lee and Tsai [14], Tsai, Lee and Ketter [15], and Tsai, Lee and Yeh [16] developed an efficient time-domain semi-analytical method to express the radiation condition in the far field region of the fluid domain. Lee and Tsai [14] derived a time-domain exact solution taking into consideration the radiation condition of the fluid domain and the deformation of the structure by using the Laplace transform method. They examined the transient analysis of the forced response of beam–water systems excited by sground accelerations in the upstream–downstream horizontal direction. A zero free surface disturbance condition was assumed, but they did not discuss the natural vibration characteristic behaviour of the systems. However, they presented a selection of free vibration results of the water domain, which are not necessarily the fundamental characteristics of the fluid–structure interaction systems, and then they used this information to evaluate the hydrodynamic pressure.

From the viewpoint of continuum mechanics, it is necessary for the fluid–structure interaction system to be considered as a total dynamical system within the dynamic analysis. Therefore, as discussed by Xing and Price [17] and Xing, Price and Du [18], there exist natural vibration characteristics (i.e., frequencies and mode shapes) and these depend on the assumptions inherent in the mathematical model (i.e., rigid or flexible structure) and the boundary conditions imposed on the structure (pinned, rigid, etc.), the free surface disturbance and the boundary conditions at infinity in the fluid domain. Xing *et al*. presented a selection of numerical results for a wide range of fluid–structure interaction systems and showed the importance of the natural vibration analysis component within the overall dynamic analysis to determine forced motion responses. In this paper, therefore, attention is focused on exact natural vibration solutions for a unified coupled beam–water dynamic system subject to a variety of assumptions and boundary conditions in order to assess the influence of such effects on solution; namely, a dry versus wet beam and the implication of flexibility on solution, the imposition of a zero surface wave disturbance or allowing waves to generate on the free surface and the imposition of an undisturbed condition at infinity in the fluid domain. An exact solution for each case is obtained using the separation of variables method (see reference [19]) and/or techniques used by others in previous studies (see, for example, references [6, 7, 11, 14]). The numerical examples presented provide a comparison of the effects of the free surface wave disturbance and fluid compressibility on the natural dynamic characteristics of the interacting system subject to the undisturbed condition at infinity. These highlight the importance of each effect in relation to the natural frequencies of the coupled fluid–beam dynamic interacting system.

2. GOVERNING EQUATIONS

Consider the beam–water interacting system as illustrated in Figure 1. Here x and y represent a two-dimensional Cartesian co-ordinate system with origin o at the intersection of the central line of the beam and horizontal floor of the reservoir. It is assumed that the water is compressible, inviscid, its motion irrotational and the reservoir is of mean depth h; the flexible uniform beam is of height H ($>h$), wet height h, of breadth F and of unit thickness perpendicular to the o–xy plane. The bending stiffness and mass density of the beam are denoted by EJ and ρ_s respectively; ρ_f and c represent the mass density and the sound speed of the water. Under the assumption of small disturbances, the linearized equations describing the dynamic pressure $p(x, y, t)$ in the water, the deflections $u_1(y, t)$, $(0 < y < h)$ and $u_2(y, t)$ $(h < y < H)$ in the beam are as follows.

2.1. FLUID DOMAIN

2.1.1. *Dynamic equation*

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 = (1/c^2)\partial^2 p/\partial t^2, \qquad 0 < x < \infty, \quad 0 < y < h. \tag{1}$$

2.1.2. *Boundary conditions*

On the *free surface*, this condition takes one of the two following forms: (1) zero free surface wave disturbance,

$$p = 0, \qquad y = h, \tag{2}$$

or (2) a free surface wave disturbance governed by the equation

$$\partial p/\partial y = -(1/g)\partial^2 p/\partial t^2, \qquad y = h. \tag{3}$$

On the *bottom* of the reservoir, assumed to be impermeable and rigid,

$$\partial p/\partial y = 0, \qquad y = 0. \tag{4}$$

At *infinity* in the water domain, it is assumed that the pressure disturbance in the water does not transmit to infinity in the water domain; namely, there is an undisturbed condition governed by the following equation

$$p = 0, \qquad x \to \infty. \tag{5}$$

2.2. SOLID DOMAIN

2.2.1. *Dynamic equation*

The equation of motion governing the submerged beam, treated for simplicity as a Bernoulli beam, is

$$EJ(\partial^4 u_1/\partial y^4) + \rho_s F(\partial^2 u_1/\partial t^2) = -p(0, y, t), \qquad 0 < y < h, \tag{6}$$

and, for the dry portion in air,

$$EJ(\partial^4 u_2/\partial y^4) + \rho_s F(\partial^2 u_2/\partial t^2) = 0, \qquad h < y < H. \tag{7}$$

2.2.2. *Boundary conditions*

At the base of the beam, assumed to be fixed,

$$u_1(0, t) = 0, \qquad (\partial u_1/\partial y)(0, t) = 0, \tag{8}$$

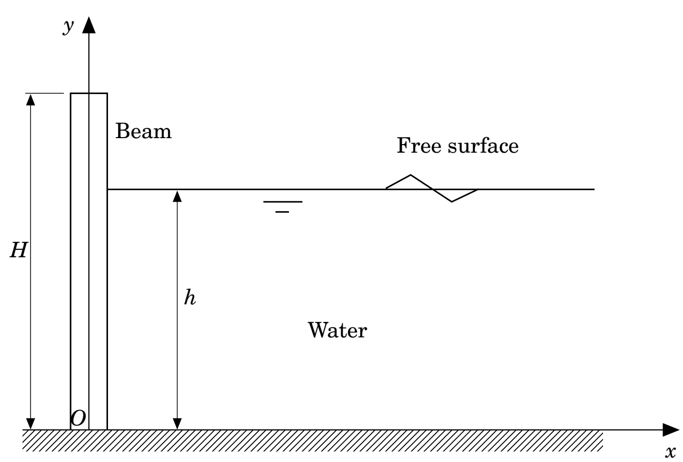

Figure 1. The coupled beam–water interaction system.

while at the free end

$$(\partial^3 u_2/\partial y^3)(H, t) = 0, \qquad (\partial^2 u_2/\partial y^2)(H, t) = 0. \tag{9}$$

On the interface between the wetted and the dry portions of the beam, the deflection, the rotation angle, the internal shear force and bending moment of the beam must be continuous. This is satisfied when

$$u_1(h, t) = u_2(h, t), \qquad (\partial u_1/\partial y)(h, t) = (\partial u_2/\partial y)(h, t), \tag{10}$$

$$(\partial^2 u_1/\partial y^2)(h, t) = (\partial^2 u_2/\partial y^2)(h, t), \qquad (\partial^3 u_1/\partial y^3)(h, t) = (\partial^3 u_2/\partial y^3)(h, t). \tag{11}$$

2.3. FLUID–STRUCTURE INTERACTION INTERFACE

On the fluid–structure interaction interface, the pressure p in the water and the displacement u_1 of the wetted beam section satisfy the relation

$$\partial p/\partial x = -\rho_f(\partial^2 u_1/\partial t^2), \qquad x = 0, \quad 0 < y < h. \tag{12}$$

3. VARIABLES SEPARABLE FORMS OF GOVERNING EQUATIONS

By using the separation of variables method (see, for example, reference [19]), solutions of the pressure p and of the displacements u_1 and u_2 are sought in the forms

$$p(x, y, t) = P(x, y)T(t) = X(x)Y(y)T(t),$$

$$u_1(y, t) = U_1(y)T(t), \qquad u_2(y, t) = U_2(y)T(t). \tag{13}$$

The substitution of these expressions into the governing equations (1)–(12) allows separation of variables and it can be shown that each variable $X(x)$, $Y(y)$, $T(t)$, $U_1(y)$ and $U_2(y)$ satisfies, respectively, the following equations.

(1) For the time function $T(t)$,

$$T'' + \hat{\Omega}^2 T = 0; \tag{14}$$

(2) for the spatial y-function $Y(y)$,

$$Y'' + \hat{\kappa}^2 Y = 0, \qquad Y'(0) = 0, \tag{15, 16}$$

subject to the boundary condition derived from equation (2)

$$Y(h) = 0 \qquad \text{when free surface waves are neglected,} \tag{17}$$

or, by (3),

$$Y' - (\hat{\Omega}^2/g)Y = 0, \quad y = h, \qquad \text{when free surface waves are included;} \tag{18}$$

(3) for the spatial x-function $X(x)$,

$$X'' + \hat{\lambda}^2 X = 0, \tag{19}$$

subject to the boundary condition (5)

$$X(x) = 0, \quad x \to \infty, \qquad \text{undisturbed case;} \tag{20}$$

(4) for the displacement functions $U_1(y)$ and $U_2(y)$:

$$EJU_1^{(4)} - \rho_s F\hat{\Omega}^2 U_1 = -X(0)Y(y), \qquad 0 < y < h, \tag{21}$$

subject to the boundary conditions (8)

$$U_1(0) = 0, \qquad U_1'(0) = 0; \tag{22, 23}$$

and

$$EJU_2^{(4)} - \rho_s F\hat{\Omega}^2 U_2 = 0, \qquad h < y < H, \tag{24}$$

subject to the boundary conditions (9)

$$U_2''(H) = 0, \qquad U_2'''(H) = 0 \tag{25, 26}$$

and the interface conditions (10) and (11)

$$U_1(h) = U_2(h), \qquad U_1'(h) = U_2'(h), \tag{27}$$

$$U_1''(h) = U_2''(h), \qquad U_1'''(h) = U_2'''(h), \tag{28}$$

and (5) on the fluid–solid interaction interface (12)

$$\rho_f\hat{\Omega}^2 U_1(y) = X'(0)Y(y), \qquad 0 < y < h. \tag{29}$$

In these equations, $()'$, $()''$, $()'''$ and $()^{(4)}$ indicate the differential order of the function $()$; $\hat{\Omega}^2$, $\hat{\kappa}^2$ and $\hat{\lambda}^2$ represent three real parameters (see the Appendix) to be determined. They satisfy the relation

$$\hat{\lambda}^2 = (\hat{\Omega}^2/c^2) - \hat{\kappa}^2. \tag{30}$$

Therefore it is assumed that the parameters $\hat{\Omega}$, $\hat{\kappa}$ and $\hat{\lambda}$ can be zero, a positive real number or a positive purely imaginary number. In the latter case, they can be rewritten as $\hat{\Omega} = \mathrm{i}\Omega$, $\hat{\kappa} = \mathrm{i}\kappa$ and $\hat{\lambda} = \mathrm{i}\lambda$, where Ω, κ and λ represent three positive real numbers. The adoption of these parameters allows the general solutions of the previous sets of equations, subject to the imposed boundary conditions, to be represented as follows.

(i) For the time function, two solutions are possible depending on the value of $\hat{\Omega}$. That is,

$$T(t) = At + B, \qquad \hat{\Omega} = 0, \tag{31}$$

$$T(t) = a\,\mathrm{e}^{\mathrm{i}\hat{\Omega}t} + b\,\mathrm{e}^{-\mathrm{i}\hat{\Omega}t}, \qquad \hat{\Omega} \neq 0. \tag{32}$$

(ii) The function $Y(y)$ satisfying equations (15) and (16) takes the form

$$Y(y) = D\cos(\hat{\kappa}y). \tag{33}$$

(iii) The solutions of the function $X(x)$ satisfying equation (19) take the forms

$$X(x) = Qx + S, \qquad \hat{\lambda} = 0, \tag{34}$$

$$X(x) = q\,\mathrm{e}^{\mathrm{i}\hat{\lambda}x} + s\,\mathrm{e}^{-\mathrm{i}\hat{\lambda}x}, \qquad \hat{\lambda} \neq 0. \tag{35}$$

Here, A, B, D, Q, S, a, b, q and s represent constants to be determined depending on the boundary conditions. The constants A, B, D, Q and S are real valued, whereas a, b, q and s may take complex valued representations.

The well known relations

$$\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x, \qquad \cos \mathrm{i}x = \cosh x, \qquad \sin \mathrm{i}x = \mathrm{i}\sinh x,$$
$$\cosh \mathrm{i}x = \cos x, \qquad \sinh \mathrm{i}x = \mathrm{i}\sin x \tag{36}$$

allow the complex forms of the functions $T(t)$, $Y(y)$ or $X(x)$ to be transformed into other forms. For example, the function $T(t)$ in equation (32) and the function $X(x)$ in equation (35) can be expressed alternatively in the forms

$$T(t) = a\cos(\hat{\Omega}t) + b\sin(\hat{\Omega}t), \qquad \text{for } \hat{\Omega}^2 \neq 0, \tag{37}$$

$$X(x) = q\cos(\hat{\lambda}x) + s\sin(\hat{\lambda}x), \qquad \text{for } \hat{\lambda}^2 \neq 0. \tag{38}$$

4. SOLUTIONS FOR THE FUNCTIONS $X(x)$, $Y(y)$ AND $T(t)$

Depending on the boundary conditions chosen, there exist two different solutions of the problem. Their combinations are as follows: (i) zero disturbance at infinity ($x \to \infty$) and free surface disturbance neglected; (ii) zero disturbance at infinity and free surface disturbance assumed. These two cases are examined in sections 4.1 and 4.2, respectively. No assumption is introduced concerning fluid compressibility (i.e., $0 < c \leqslant \infty$).

From equations (20), (34) and (35), it is found that

$$X(x) \equiv 0, \qquad \hat{\lambda} = 0, \quad \hat{\lambda} > 0, \tag{39}$$

$$X(x) = \mathrm{e}^{-\lambda x}, \qquad \hat{\lambda} = \mathrm{i}\lambda. \tag{40}$$

Therefore, a non-trivial solution of the problem is obtained only if $\hat{\lambda} = \mathrm{i}\lambda$.

4.1. FREE SURFACE WAVE NEGLECTED

The function $Y(y)$ expressed in equations (33) must satisfy the boundary conditions (17), from which it follows that

$$\cos(\hat{\kappa}h) = 0. \tag{41}$$

Solving this equation yields the result

$$Y(y) \equiv 0, \qquad \text{for } \hat{\kappa} = 0 \quad \text{or} \quad \hat{\kappa} = \mathrm{i}\kappa, \tag{42}$$

$$Y_n(y) = \cos(\hat{\kappa}_n y), \qquad \text{for } \hat{\kappa}_n = (2n-1)\pi/2h, \qquad n = 1, 2, 3, \ldots. \tag{43}$$

In combination with equations (30) and (40), for each $\hat{\Omega}$ the non-trivial solutions for the functions of $Y(y)$ and $X(x)$ corresponding to $T(t)$ in equations (31) and (32) are

$$Y_n(y) = \cos(\hat{\kappa}_n y), \qquad X_n(x) = \mathrm{e}^{-\lambda_n x}, \tag{44}$$

$$\hat{\kappa}_n = (2n-1)\pi/2h, \quad n = 1, 2, 3, \ldots, \qquad \lambda_n^2 = \hat{\kappa}_n^2 - \hat{\Omega}^2/c^2 > 0, \quad \text{for arbitrary } \hat{\Omega}.$$

4.2. FREE SURFACE WAVE INCLUDED

The function $Y(y)$ in equation (33) satisfies the boundary condition (18), from which it follows that

$$\tan(\hat{\kappa}h) = -\hat{\Omega}^2/g\hat{\kappa}. \tag{45}$$

The solution of this equation allows $Y(y)$ to be expressed in the forms

$$Y(y) = d, \qquad \hat{\kappa} = 0 = \hat{\Omega}, \tag{46}$$

$$Y_0(y) = \cos(\hat{\kappa}_0 y) = \cosh(\kappa_0 y), \qquad \hat{\kappa}_0 = \mathrm{i}\kappa_0, \quad \hat{\Omega} > 0, \tag{47}$$

$$Y_n(y) = \cos(\hat{\kappa}_n y), \qquad \hat{\kappa}_n > 0, \quad n = 1, 2, 3, \ldots, \quad \text{for arbitrary } \hat{\Omega}. \tag{48}$$

Here $\hat{\kappa}_n$ ($n = 0, 1, 2, 3, \ldots$) is derived as follows.

The first solution κ_0 of the equation

$$\tanh(\kappa_0 h) = \hat{\Omega}^2/g\kappa_0, \tag{49}$$

is shown in Figure 2.

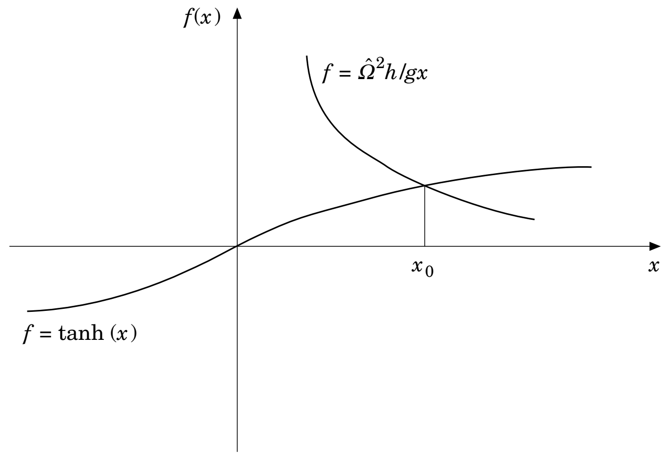

Figure 2. The solution κ_0 of equation (49) corresponding to a particular value $\hat{\Omega}$ ($x = \kappa h$, $x_0 = \kappa_0 h$).

By solving equation (45), the remaining values $\hat{\kappa}_n$ ($n = 1, 2, 3, \ldots$) are obtained as follows:

$$\hat{\kappa}_n h = n\pi, \qquad \text{for } \hat{\Omega} = 0, \tag{50}$$

$$\hat{\kappa}_n h \in ((n-1)\pi, (2n-1)\pi/2), \qquad \text{for } \hat{\Omega} = i\Omega, \tag{51}$$

$$\hat{\kappa}_n h \in ((2n-1)\pi/2, n\pi), \qquad \text{for } \hat{\Omega} > 0. \tag{52}$$

The solutions given in equations (51) and (52) can be derived numerically or in a graphical manner as illustrated in Figures 3 and 4.

In combination with equations (30) and (40), for each $\hat{\Omega}$ the non-trivial solutions of the functions $Y(y)$ and $X(x)$ corresponding to $T(t)$ in equations (31) and (32) are given by

$$Y_n(y) = \cos(\hat{\kappa}_n y), \qquad X_n(x) = e^{-\lambda_n x}, \qquad \lambda_n^2 = \hat{\kappa}_n^2 - \hat{\Omega}^2/c^2 > 0, \tag{53}$$

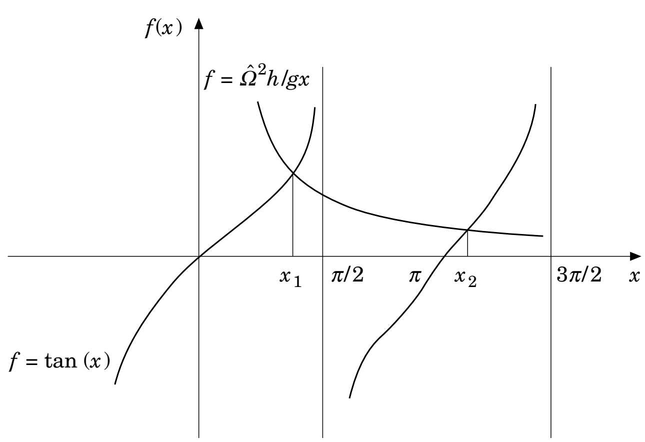

Figure 3. A series of solution $\hat{\kappa}_n$ of equation (51) corresponding to each Ω ($x = \hat{\kappa}h$, $x_1 = \hat{\kappa}_1 h$, $x_2 = \hat{\kappa}_2 h$).

$$\tan(\hat{\kappa}_n h) = -\frac{\hat{\Omega}^2}{g\hat{\kappa}_n}, \qquad \hat{\kappa}_n : \begin{cases} = \dfrac{n\pi}{h} & \text{if } \hat{\Omega} = 0 \\ \in \left(\dfrac{(n-1)\pi}{h}, \dfrac{(2n-1)\pi}{2h}\right) & \text{if } \hat{\Omega} = \mathrm{i}\Omega \\ \in \left(\dfrac{(2n-1)\pi}{2h}, \dfrac{n\pi}{h}\right) & \text{if } \hat{\Omega} > 0, \quad n = 1, 2, 3, \ldots \end{cases}.$$

5. EIGENVALUE EQUATIONS

The eigenvalue equations of the beam–water system can be derived by using equations (21)–(29). For the case $\hat{\Omega}^2 < 0$, $\hat{\Omega} = \mathrm{i}\Omega$, the exponential solutions of the function $T(t)$ do not represent natural vibration solutions, so that only the cases $\hat{\Omega}^2 \geqslant 0$ need be examined. The case $\hat{\Omega}^2 = 0$ provides a static solution equivalent to rigid mode solutions in structural vibration. However, for the undisturbed condition at infinity in the fluid domain there exists no non-trivial static solution associated with $\hat{\Omega}^2 = 0$.

For a positive value of $\hat{\Omega}$, the function $T(t)$ given in equation (32) is complex and the real pressure $p(x, y, t)$, displacements $u_1(y, t)$ and $u_2(y, t)$ defined in equation (13) are represented by their corresponding complex functions. It must be noted that only the real parts of these complex functions p, u_1 and u_2 represent real physical quantities (see, for example, reference [20]).

The solutions of the functions $Y(y)$ and $X(x)$ are given by equation (44) for the case in which free surface waves are ignored and equation (53) in the presence of a free surface wave disturbance.

During the calculation to obtain the curve of $|\mathbf{R}| \sim \omega$ in Figure 5, for each given value of $\hat{\Omega}$, there exists a series of functions $Y_n(y)$ and $X_n(x)$. Let n_1 denote a minimum positive integer for which the inequality $\hat{\kappa}_n^2 > \hat{\Omega}^2/c^2$ is satisfied. By adopting the superposition principle, it can be shown that

$$X(x)Y(y) = \sum_{n=n_1}^{\infty} G_n \, \mathrm{e}^{-\lambda_n x} \cos(\hat{\kappa}_n y), \tag{54}$$

where each unknown G_n is a real constant.

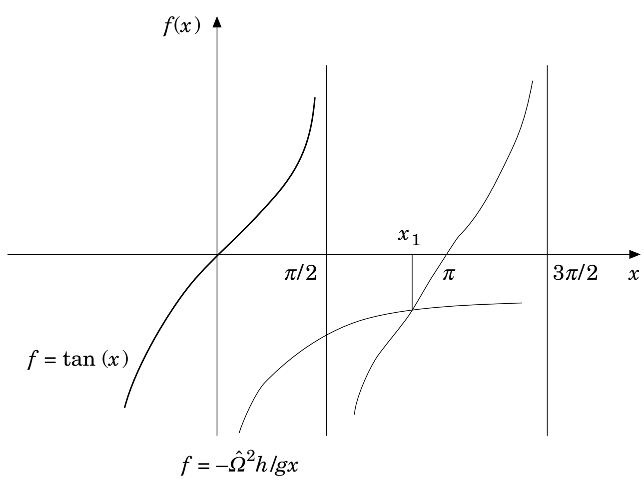

Figure 4. A series of solution $\hat{\kappa}_n$ of equation (52) corresponding to each $\hat{\Omega}$ ($x = \hat{\kappa}h$, $x_1 = \hat{\kappa}_1 h$).

For a more general discussion of the proposed solutions, one can define the following non-dimensional parameters:

$$\xi = y/H, \qquad v = h/H, \qquad \gamma = \rho_f H/\rho_s F, \qquad \omega^2 = \hat{\Omega}/\Omega_b, \qquad \bar{\kappa}_n = \hat{\kappa}_n H,$$
$$\bar{\lambda}_n^2 = \bar{\kappa}_n^2 - \omega^4/\bar{c}^2, \qquad \bar{c} = c/\Omega_b H. \tag{55}$$

Here $\Omega_b = [EJ/(\rho_s F H^4)]^{1/2}$ represents a frequency parameter of the dry beam. The variables ξ and v denote a non-dimensional co-ordinate and the ratio of water depth to beam height, respectively; γ is the mass ratio of water to beam and ω^2 denotes the frequency parameter.

On substituting equations (54) and (55) into equations (21), (24) and (29), one finds that

$$\bar{U}_1^{(4)}(\xi) - \omega^4 \bar{U}_1(\xi) = -\sum_{n=n_1}^{\infty} A_n \cos(\bar{\kappa}_n \xi), \qquad 0 < \xi < v, \tag{56}$$

$$\bar{U}_2^{(4)}(\xi) - \omega^4 \bar{U}_2(\xi) = 0, \qquad v < \xi < 1, \tag{57}$$

$$\bar{U}_1(\xi) = -\sum_{n=n_1}^{\infty} \frac{\bar{\lambda}_n A_n}{\gamma\omega^4} \cos(\bar{\kappa}_n \xi), \qquad 0 < \xi < v, \tag{58}$$

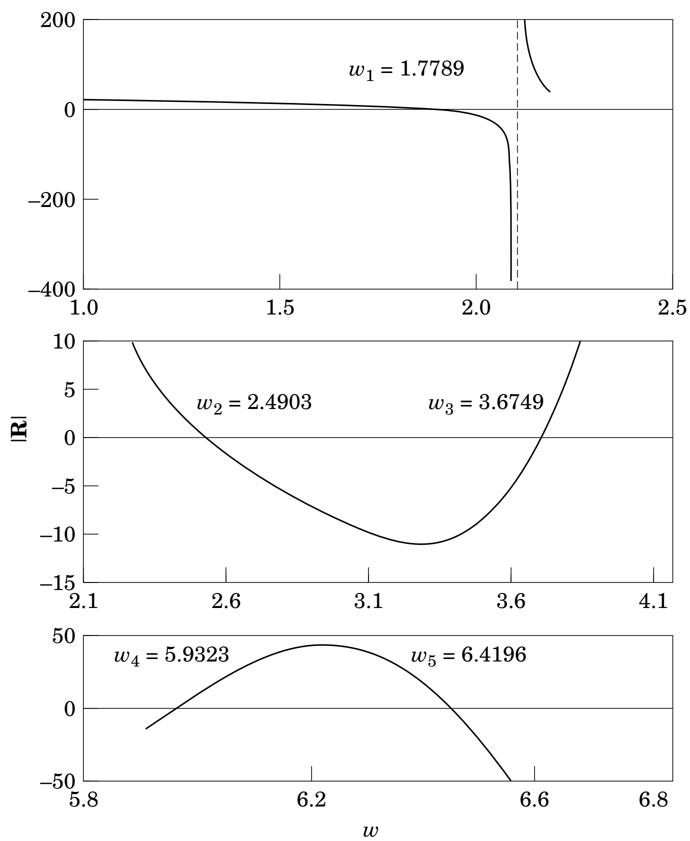

Figure 5. A typical curve of $|\mathbf{R}| \sim \omega$ used to determine the natural frequencies of the beam–water interaction system.

where each $A_n = G_n H^3/EJ$ is a non-dimensional real constant for $n_1 \leqslant n \leqslant \infty$. The solution $\bar{U}_1(\xi)$ satisfying equation (56) and the solution $\bar{U}_2(\xi)$ satisfying equation (57) are expressible in the series forms

$$\bar{U}_1 = \sum_{j=1}^{4} D_j \phi_j(\xi) + \sum_{n=n_1}^{\infty} B_n \cos(\bar{\kappa}_n \xi), \qquad \bar{U}_2 = \sum_{j=5}^{8} D_j \phi_j(\xi), \tag{59}$$

where,

$$B_n = -A_n/(\bar{\kappa}_n^4 - \omega^4), \tag{60}$$

and the real valued beam functions ϕ_m $(1 \leqslant m \leqslant 8)$ are defined as follows:

$$\phi_1(\xi) = \cos(\omega\xi), \qquad \phi_2(\xi) = \sin(\omega\xi), \qquad \phi_3(\xi) = \cosh(\omega\xi), \qquad \phi_4(\xi) = \sinh(\omega\xi),$$

$$\phi_5(\xi) = \cos[\omega(\xi - 1)], \qquad \phi_6(\xi) = \sin[\omega(\xi - 1)],$$

$$\phi_7(\xi) = \cosh[\omega(\xi - 1)], \qquad \phi_8(\xi) = \sinh[\omega(\xi - 1)]. \tag{61}$$

On substituting equation (59) into equation (58) and using the orthogonality relation of the functions $Y_n(\xi)$,

$$\int_0^v Y_n(\xi) Y_m(\xi)\, \mathrm{d}\xi = \left\{\begin{matrix} 0, & m \neq n \\ \dfrac{v}{2} + \dfrac{\sin(2\bar{\kappa}_n v)}{4\bar{\kappa}_n} = I_n, & m = n \end{matrix}\right\}, \tag{62}$$

the following results are obtained:

$$A_n = E_n \sum_{j=1}^{4} D_j I_{nj}, \qquad B_n = \tilde{E}_n \sum_{j=1}^{4} D_j I_{nj}, \tag{63}$$

where

$$E_n = 1\bigg/\left\{ I_n \left[\frac{1}{\bar{\kappa}_n^4 - \omega^4} - \frac{\bar{\lambda}_n}{\gamma\omega^4} \right] \right\}, \qquad \tilde{E}_n = -1\bigg/\left\{ I_n \left[\frac{1}{\bar{\kappa}_n^4 - \omega^4} - \frac{\bar{\lambda}_n(\bar{\kappa}_n^4 - \omega^4)}{\gamma\omega^4} \right] \right\},$$

$$I_{n1} = \int_0^v \cos(\bar{\kappa}_n \xi) \phi_1(\xi)\, \mathrm{d}\xi, \qquad I_{n2} = \int_0^v \cos(\bar{\kappa}_n \xi) \phi_2(\xi)\, \mathrm{d}\xi,$$

$$I_{n3} = \int_0^v \cos(\bar{\kappa}_n \xi) \phi_3(\xi)\, \mathrm{d}\xi, \qquad I_{n4} = \int_0^v \cos(\bar{\kappa}_n \xi) \phi_4(\xi)\, \mathrm{d}\xi. \tag{64}$$

The substitution of equations (59) and (60) into the non-dimensional forms of equations (22), (23) and (25)–(28) produces a linear homogeneous system of algebraic equations which can be written in the matrix form

$$\mathbf{RD} = \mathbf{0}. \tag{65}$$

In this equation, $\mathbf{D}$ represents a vector

$$\mathbf{D}^{\mathrm{T}} = [D_1 \quad D_2 \quad \cdots \quad D_8], \tag{66}$$

and **R** represents a 8×8 square matrix. The non-zero elements of this matrix are given by

$$R_{1j} = \phi_j(0) + \sum_{n=n_1}^{\infty} \tilde{E}_n I_{nj}, \quad R_{2j} = \phi_j'(0), \qquad j = 1, 2, 3, 4,$$

$$R_{3j} = \phi_j''(1), \quad R_{4j} = \phi_j'''(1), \qquad j = 5, 6, 7, 8,$$

$$R_{5j} = \begin{Bmatrix} \phi_j(v) + \sum_{n=n_1}^{\infty} \tilde{E}_n I_{nj} Y_n(v), & j = 1, 2, 3, 4 \\ -\phi_j(v), & j = 5, 6, 7, 8 \end{Bmatrix},$$

$$R_{6j} = \begin{Bmatrix} \omega\phi_j'(v) + \sum_{n=n_1}^{\infty} \tilde{E}_n I_{nj} Y_n'(v), & j = 1, 2, 3, 4 \\ -\omega\phi_j'(v), & j = 5, 6, 7, 8 \end{Bmatrix},$$

$$R_{7j} = \begin{Bmatrix} \omega^2\phi_j''(v) + \sum_{n=n_1}^{\infty} \tilde{E}_n I_{nj} Y_n''(v), & j = 1, 2, 3, 4 \\ -\omega^2\phi_j''(v), & j = 5, 6, 7, 8 \end{Bmatrix},$$

$$R_{8j} = \begin{Bmatrix} \omega^3\phi_j'''(v) + \sum_{n=n_1}^{\infty} \tilde{E}_n I_{nj} Y_n'''(v), & j = 1, 2, 3, 4 \\ -\omega^3\phi_j'''(v), & j = 5, 6, 7, 8 \end{Bmatrix}. \tag{67}$$

Therefore, the characteristic eigenvalue equation of the beam–water system is

$$|\mathbf{R}| = 0, \tag{68}$$

from which the natural frequency parameters ω can be determined. The constant vector **D** denoted in equations (66) can be obtained by solving equation (65). For each natural frequency parameter ω, the vibration forms $\bar{U}_1(\xi)$ and $\bar{U}_2(\xi)$ for the beam and $X(x)Y(y)$ for the water pressure p can be calculated through equations (59) and (54). These natural vibration forms satisfy orthogonality relations, as demonstrated in Appendix A.

4. NUMERICAL RESULTS

By using the theoretical formulations derived in the previous sections, numerical results describing the dynamical behaviour of beam–water systems were obtained. In these calculations, it was assumed that the density of the beam and the density of water are $\rho_s = 2{\cdot}4 \times 10^3$ kg/m^3 and $\rho_f = 1{\cdot}0 \times 10^3$ kg/m^3, respectively; the elastic modulus of the beam is $E = 2{\cdot}94 \times 10^{10}$ Pa; the speed of sound in water is $c = 1439$ m/s.

The natural frequencies of the coupled system are determined through the solutions of equation (65). For free surface waves neglected, the parameters $\hat{\kappa}_n$ given in equation (44) are independent of the natural frequency $\hat{\Omega}$ and the latter can be determined by solving equation (65) only. However, when free surface waves are present, the parameters $\hat{\kappa}_n$ determined by equation (53) are dependent on the natural frequency $\hat{\Omega}$ and it is therefore necessary to solve equations (65) and (53) for these frequency values. A typical curve

TABLE 1

The first frequency parameter ω_1

	γ						
v	0·0	0·5	1·0	3·0	5·0	8·0	10·0
0·0	1·8746						
0·5		1·8737	1·8725	1·8673	1·8620	1·8544	1·8493
0·8		1·8592	1·8419	1·7825	1·7311	1·6654	1·6275
1·0		1·8230	1·7773	1·6375	1·5391	1·4340	1·3801

TABLE 2

The second frequency parameter ω_2

	γ						
v	0·0	0·5	1·0	3·0	5·0	8·0	10·0
0·0	4·4810						
0·5		4·6563	4·6198	4·4847	4·3657	4·2130	4·1253
0·8		4·5591	4·4470	4·1362	3·9427	3·7522	3·6610
1·0		4·5276	4·3962	4·0450	3·8224	3·6770	3·5450

illustrating the variation of the value of the determinant $|\mathbf{R}|$ with the frequency parameter ω is shown in Figure 5. The points at which the determinant $|\mathbf{R}|$ has zero values denote natural frequencies: i.e., 1·7789, 2·4903, 3·6749, 5·9323 and 6·4196. These first five natural frequencies are the values of ω^{cs} in Table 4 corresponding to the case of a beam–water system, a free surface wave disturbance (s) and fluid compressibility (c).

6.1. INCOMPRESSIBLE WATER WITH NO FREE SURFACE

The speed of sound in an incompressible fluid tends to infinity, so that formulations suitable for an incompressible fluid are deduced by substituting $1/c \to 0$ into the equations presented in previous sections. As obtained by this approach, the first three frequency parameters calculated are listed in Tables 1–3.

A comparison of these results indicates that the effect of water on the beam provides an influence similar to the inclusion of an additional mass or a fluid action in phase with an inertia force. The frequency corresponding to each mode of the beam–water system decreases with increasing ratio v/γ. This conclusion re-affirms the mathematically proven results of Xing and Price [17].

TABLE 3

The third frequency parameter ω_3

	γ						
v	0·0	0·5	1·0	3·0	5·0	8·0	10·0
0·0	7·8540						
0·5		7·7509	7·6583	7·3728	7·1753	6·9691	6·8671
0·8		7·7007	7·5704	7·1792	6·8989	6·5849	6·4197
1·0		7·6874	7·5450	7·1148	6·8048	6·4562	6·2712

TABLE 4

The frequency parameters of the long, thin beam–water system ($v = 0{\cdot}8$, $\gamma = 10{\cdot}0$)

Dry beam, ω^a	Rigid beam		Wet flexible beam			
	ω^c	ω^{cs}	ω^i	ω^c	ω^{is}	ω^{cs}
1·8746			1·6275	1·6275	1·7789	1·7789
	1·9635	2·4900	1·9633	1·9633	2·4903	2·4903
4·6810			3·6610	3·6603	3·6755	3·6749
	5·8905	5·9319	5·8905	5·8905	5·9323	5·9323
7·8540			6·4197	6·4153	6·4241	6·4196

6.2. LONG THIN AND SHORT THICK BEAMS

To compare the natural characteristics of beam–water systems, by way of example, two systems, $v = 0{\cdot}8$ and $\gamma = 10{\cdot}0$, as well as $v = 0{\cdot}8$ and $\gamma = 0{\cdot}5$, are examined. These imply that for the fixed ratio $v = 0{\cdot}8$, i.e., depth of water to length of beam, the ratio H/F, representing length to width of beam, changes from $0{\cdot}5\rho_s/\rho_f$ to $10\rho_s/\rho_f$. That is, for a fixed value of ρ_s/ρ_f the former value represents a short, thick beam whereas the latter represents a long, thin beam. In particular, for $\gamma = 0{\cdot}5$, it follows that $H/F = 1{\cdot}2$ for a beam of density $\rho_s = 2{\cdot}4 \times 10^3\ \mathrm{kg/m^3}$ ($\rho_f = 1{\cdot}0 \times 10^3\ \mathrm{kg/m^3}$) and $H/F = 3{\cdot}6$ for a steel beam with $\rho_s = 7{\cdot}8 \times 10^3\ \mathrm{kg/m^3}$. In the short, thick beam example, it is noted that the beam bending theory is not strictly applicable and a theory incorporating shear effects or a more accurate beam theory is needed. However, from the viewpoint of an engineering approximation, a simple beam bending theory can give a first order approximation to the solution of the dynamical problem. For example, results obtained by a two-dimensional analysis for a dam of height 160 m, base width 120 m and top width 16 m demonstrated that the first two important vibration forms are those associated with beam bending. The natural frequency parameters for the long, thin and short, thick beams are given in Tables 4 and 5, respectively. The distribution of displacement and dynamic pressure along the beam loaded by water corresponding to the first natural frequency are shown in Figures 6 and 7. Here, superscripts i and c denote incompressible and compressible water assumptions; a represents the case of a dry beam, i.e., the beam in air; s represents the case in which the influence of a free surface wave disturbance is examined. For example, ω^c denotes the frequency parameter of the fluid assuming compressibility and zero disturbance on the free surface, whereas ω^{is} denotes the frequency parameter when the free surface wave effect is included and fluid incompressibility assumed, etc.

TABLE 5

The frequency parameters of the short, thick beam–water system ($v = 0{\cdot}8$, $\gamma = 10{\cdot}0$)

Dry beam, ω^a	Rigid beam		Wet flexible beam			
	ω^c	ω^{cs}	ω^i	ω^c	ω^{is}	ω^{cs}
1·8746			1·8592	1·8570	1·8590	1·8570
	1·9635	2·4900	1·9634	1·9634	2·4901	2·4901
4·6810			4·5591	4·5587	4·5590	4·5590
	5·8905	5·9319	5·8905	5·8905	5·9317	5·9317
7·8540			7·7007	7·6974	7·7009	7·6976

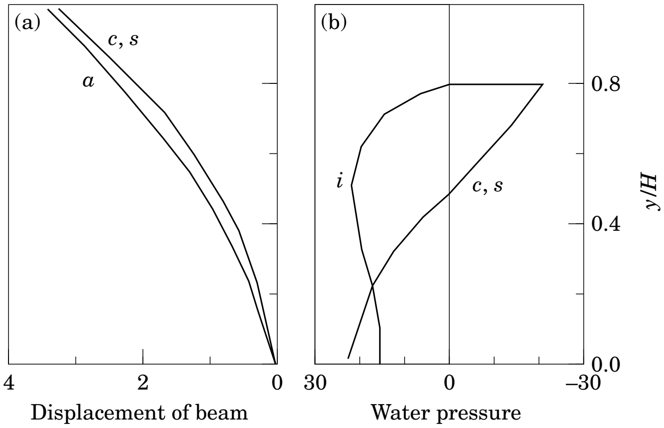

Figure 6. The first vibration forms of the long, thin beam–water interaction system: (a) the vibration form of the beam; (b) the vibration form of the dynamic water pressure along the beam.

A comparison of these results shows the following.

(i) For an assumed rigid beam, the free surface wave effect causes an increase in the value of the natural frequencies of the fluid domain. This is due to the influence of the potential associated with the free surface disturbance.

(ii) The number of natural frequencies of the coupled system equals the sum of the number of natural frequencies for the dry elastic beam and the number of natural frequencies for the water domain in the rigid beam case. This suggests that the interaction does not change the number of natural frequencies of the total coupled system.

(iii) For an assumed flexible beam, the calculated natural frequencies (corresponding to frequencies of the beam) of the coupled system are lower than the natural frequencies of the dry beam. However, the calculated natural frequencies (corresponding to frequencies of the water domain) of the coupled system are not obviously changed. Further calculations indicate that for a fixed value of v, an increase in v causes a decrease in the values of the natural frequencies. Therefore, the effect of water in the beam–water system

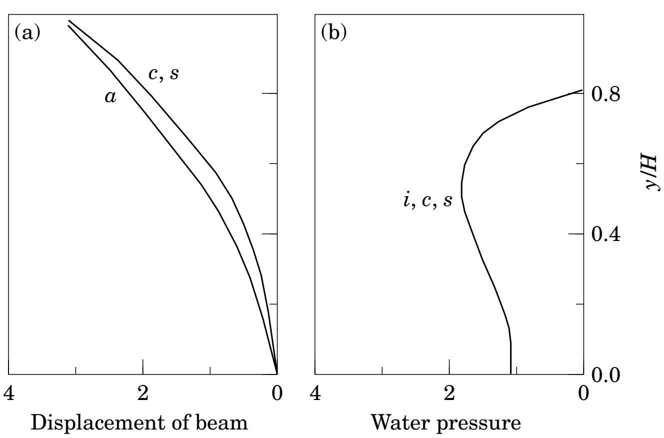

Figure 7. The first vibration forms of the short, thick beam–water interaction system: (a) the vibration form of the beam; (b) the vibration form of the dynamic water pressure along the beam.

is equivalent to an additional mass attached to the beam (or fluid action in phase with the inertia force).

(iv) The free surface wave disturbance plays a more dominant role in influencing the dynamical behaviour of the coupled beam water interaction system in the low frequency region, whereas fluid compressibility is more influential at higher frequencies.

7. CONCLUSIONS

The dynamical behaviour of a flexible beam–water interaction system has been studied, subject to boundary conditions at infinity (i.e., zero disturbance) in the water domain and on the free surface (i.e., zero or linear surface wave). The governing equations describing the system were analyzed using the separation of variables method. The eigenvalue equation of the natural vibration of the coupled system was derived and exact solutions obtained.

For these chosen boundary conditions, the natural vibration forms corresponding to the natural frequencies satisfy an orthogonality relation as demonstrated in Appendix A. However, when the Sommerfeld radiation condition (Appendix B) is introduced as a replacement boundary condition at infinity, this allows wave disturbances to propagate to infinity with no reflection. The corresponding eigenvalues are complex in form and hence the complex frequency is not a true natural vibration of the coupled fluid–structure interaction system in the context of this paper.

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REFERENCES

1. A. Sommerfeld 1949 *Partial Differential Equations in Physics*. New York: Academic Press.
2. H. M. Westergaard 1933 *Transactions of the American Society of Civil Engineers* **98**, 418–433. Water pressures on dams during earthquakes.
3. A. K. Chopra 1967 *Journal of Engineering Mechanics*, *American Society of Civil Engineers* **93**, 205–223. Hydrodynamic pressures on dams during earthquakes.
4. A. T. Chwang 1978 *Journal of Fluid Mechanics* **87**, 343–348. Hydrodynamic pressures on sloping dams during earthquakes, part 2: exact theory.
5. P. L.-F. Liu 1986 *Journal of Fluid Mechanics* **165**, 131–145. Hydrodynamic pressures on rigid dams during earthquakes.
6. J. T. Xing and W. G. Price 1990 *Proceedings of the International Conference on Vibration Problems in Engineering* (Q. H. Du, editor) **2**, 1001–1006. Beijing: International Academic Publishers. Hydrodynamic loadings on dams excited by horizontal and vertical vibrations.
7. Y. Zhu, Z. Weng and J. Wu 1988 *Applied Mathematics and Mechanics* **9**, 305–316. Vibration analysis of elliptical column partially submerged in water.
8. D. Zhou 1991 *Chinese Journal of Vibration Engineering* **4**, 78–82. The characteristic of free vibration of cantilever beam acted by liquid on one side (in Chinese).
9. H. Goto and K. Toki 1965 *Kyoto University—Faculty Engineering—Memoirs* **27**, 17–30. Vibrational characteristics and seismic design of submerged bridge piers.
10. C. Y. Liaw and A. K. Chopra 1974 *Earthquake Engineering and Structural Dynamics* **3**, 33–49. Dynamics of towers surrounded by water.

11. Y. ZHU, Z. WENG and J. WU 1989 *Acta Mechanics Sinica* **21**, 657–667. The coupled vibration between column and water considering the effects of surface wave and compressibility of water (in Chinese).
12. C. S. TSAI and G. C. LEE 1990 *Journal of Engineering Mechanics*, *American Society of Civil Engineers* **116**, 2151–2172. Methods for the transient analysis of three dimensional dam–reservoir interactions.
13. C. S. TSAI and G. C. LEE 1991 *Journal of Engineering Mechanics*, *American Society of Civil Engineers* **117**, 2007–2026. Time-domain analyses of dam–reservoir system, II: substructure method.
14. G. C. LEE and C. S. TSAI 1991 *Journal of Engineering Mechanics*, *American Society of Civil Engineers* **117**, 1990–2006. Time-domain analyses of dam–reservoir system, I: exact solution.
15. C. S. TSAI, G. C. LEE and R. L. KETTER 1990 *International Journal of Numerical Methods in Engineering* **29**, 913–933. A semi-analytical method for time-domain analyses of dam–reservoir interactions.
16. C. S. TSAI, G. C. LEE and C. S. YEH 1990 *Proceedings of Advances in Boundary Elements Methods in Japan and U.S.A*. (C. A. Brebbia and R. Shaw, editors) **7**, 183–199. Transient responses of 3-D fluid structure interactions.
17. J. T. XING and W. G. PRICE 1991 *Proceedings of the Royal Society of London* **A 433**, 235–255. A mixed finite element method for the dynamic analysis of coupled fluid–solid interaction problems.
18. J. T. XING, W. G. PRICE and Q. H. DU 1996 *Philosophical Transaction of the Royal Society of London* **A354**, 259–295. Mixed finite element substructure–subdomain methods for the dynamic analysis of coupled fluid–solid interaction problems.
19. R. COURANT and D. HILBERT 1962 *Methods of Mathematical Physics*. New York: Interscience.
20. A. B. PIPPARD 1978 *The Physics of Vibrations*, Volume 1. London: Cambridge University Press.

APPENDIX A: ORTHOGONALITY RELATION OF NATURAL VIBRATION FORMS

A solution is assumed in the form $p(x, y, t) = P(x, y)T(t)$, $u_1(y, t) = U_1(y)T(t)$ and $u_2(y, t) = U_2(y)T(t)$. The substitution of this solution into equations (1)–(12) gives

$$P_{,ii} + (\hat{\Omega}^2/c^2)P = 0, \qquad P_{,y}(x, 0) = 0, \tag{A1, A2}$$

$$P(x, h) = 0, \qquad \text{with the free surface wave neglected,} \tag{A3}$$

or

$$P_{,y} - (\hat{\Omega}^2/g)P = 0, \quad y = h, \qquad \text{with the free surface wave included} \tag{A4}$$

$$P(x, y) = 0, \qquad x \to \infty, \tag{A5}$$

in addition to equations (14) and (21)–(29) with $P(0, y)$ replacing $X(0)Y(y)$ and $X'(0)Y(y)$ by $P_{,x}(0, y)$. For convenience, a tensor index i $(i = 1, 2)$ is used to represent x and y, respectively, and $()_{,ii} = ()_{,11} + ()_{,22}$; $()_{,i} = \partial()/\partial x_i$. In the same way, a set of equations satisfied by the conjugate solution $p^* = P^*T^*$, $u_1^* = U_1^*T^*$ and $u_2^* = U_2^*T^*$ can be derived, which is similar to the set satisfied by the solution $p = PT$, $u_1 = U_1T$ and $u_2 = U_2T$ but with all complex quantities replaced by their corresponding conjugate quantities.

It is assumed that $\hat{\Omega}_n^2$ and $\hat{\Omega}_m^{2*}$ are two different natural frequencies and P_n, U_{1n}, U_{2n} and P_m^*, U_{1m}^*, U_{2m}^* the corresponding natural vibration forms satisfying equations (A1)–(A5) and (21)–(29) and their conjugate equations. For these solutions, it follows that

$$\frac{\hat{\Omega}_m^{2*} - \hat{\Omega}_n^2}{c^2}\int_\Gamma P_n P_m^* \,\mathrm{d}\Gamma + EJ\rho_f \int_0^h (U_{1m}^* U_{1n}^{(4)} \hat{\Omega}_m^{2*} - U_{1n} U_{1m}^{*(4)} \hat{\Omega}_n^2)\,\mathrm{d}y$$

$$+EJ\rho_f \int_h^H (U_{2m}^* U_{2n}^{(4)} \hat{\Omega}_m^{2*} - U_{2n} U_{2m}^{*(4)} \hat{\Omega}_n^2)\, \mathrm{d}y = \int_\Gamma (P_m^* P_{n,ii} - P_n P_{m,ii}^*)\, \mathrm{d}\Gamma$$

$$+\rho_f \int_0^h [P_m^*(0,y) U_{1n} \hat{\Omega}_n^2 - P_n(0,y) U_{1m}^* \hat{\Omega}_m^{2*}]\, \mathrm{d}y, \qquad \text{(A6)}$$

where Γ represents the domain of water with its boundary S. By using Green's theorem, it follows that

$$\int_\Gamma (P_m^* P_{n,ii} - P_n P_{m,ii}^*)\, \mathrm{d}\Gamma = \int_s (P_m^* P_{n,i} - P_{m,i}^* P_n)\eta_i\, \mathrm{d}S, \qquad \text{(A7)}$$

$$\int_0^h (U_{1m}^* U_{1n}^{(4)} \hat{\Omega}_m^{2*} - U_{1n} U_{1m}^{*(4)} \hat{\Omega}_n^2)\, \mathrm{d}y = (\hat{\Omega}_m^{2*} - \hat{\Omega}_n^2) \int_0^h U_{1n}'' U_{1m}^{*\prime\prime}\, \mathrm{d}y + [U_{1m} U_{1n}^{*\prime\prime\prime} - U_{1m}^{*\prime} U_{1n}''$$

$$+ U_{1m}^{*\prime\prime\prime} U_{1n} - U_{1m}^{*\prime\prime} U_{1n}']|_0^h, \qquad \text{(A8)}$$

$$\int_h^H (U_{2m}^* U_{2n}^{(4)} \hat{\Omega}_m^{2*} - U_{2n} U_{2m}^{*(4)} \hat{\Omega}_n^2)\, \mathrm{d}y = (\hat{\Omega}_m^{2*} - \hat{\Omega}_n^2) \int_h^H U_{2n}'' U_{2m}^{*\prime\prime}\, \mathrm{d}y + [U_{2m} U_{2n}^{*\prime\prime\prime} - U_{2m}^{*\prime} U_{2n}''$$

$$+ U_{2m}^{*\prime\prime\prime} U_{2n} - U_{2m}^{*\prime\prime} U_{2n}']|_h^H, \qquad \text{(A9)}$$

which, in connection with the boundary conditions expressed in equations (A2)–(A5), (22)–(23) and (25)–(28), gives

$$\int_\Gamma (P_m^* P_{n,ii} - P_n P_{m,ii}^*)\, \mathrm{d}\Gamma = -\beta \frac{\hat{\Omega}_m^{2*} - \hat{\Omega}_n^2}{g} \int_0^\infty P_n(x,h) P_m^*(x,h)\, \mathrm{d}x - J_{mn}(0) + J_{mn}(\infty), \qquad \text{(A10)}$$

$$\int_0^h (U_{1m}^* U_{1n}^{(4)} \hat{\Omega}_m^{2*} - U_{1n} U_{1m}^{*(4)} \hat{\Omega}_n^2)\, \mathrm{d}y + \int_h^H (U_{2m}^* U_{2n}^{(4)} \hat{\Omega}_m^{2*} - U_{2n} U_{2m}^{*(4)} \hat{\Omega}_n^2)\, \mathrm{d}y$$

$$= (\hat{\Omega}_m^{2*} - \hat{\Omega}_n^2)\left[\int_0^h U_{1n}'' U_{1m}^{*\prime\prime}\, \mathrm{d}y + \int_h^H U_{2n}'' U_{2m}^{*\prime\prime}\, \mathrm{d}y\right]. \qquad \text{(A11)}$$

where

$$J_{mn}(x) = \int_0^h [P_m^*(x,y) P_{n,x}(x,y) - P_{m,x}^*(x,y) P_n(x,y)\, \mathrm{d}y, \qquad \text{(A12)}$$

$$J_{mn}(\infty) = \lim_{x \to \infty} J_{mn}(x), \qquad \text{(A13)}$$

$$\beta = \begin{Bmatrix} 0 & \text{with the free surface wave neglected} \\ 1 & \text{with the free surface wave included} \end{Bmatrix}. \qquad \text{(A14)}$$

The substitution of equations (A7)–(A13) into equation (A6) and using equation (29) gives

$$(\hat{\Omega}_m^{2*} - \hat{\Omega}_n^2)\left\{\frac{1}{c^2}\int_\Gamma P_n P_m^* \,\mathrm{d}\Gamma + EJ\rho_f\left[\int_0^h U_{1n}'' U_{1m}^{*\prime\prime}\,\mathrm{d}y + \int_h^H U_{2n}'' U_{2m}^{*\prime\prime}\,\mathrm{d}y\right]\right.$$

$$\left. + \frac{\beta}{g}\int_0^\infty P_n(x,h)P_m^*(x,h)\,\mathrm{d}x\right\} = J_{mn}(\infty). \quad \text{(A15)}$$

From equation (A5) and its conjugate form in the undisturbed case, it follows that $J_{mn}(\infty) = 0$. For equation (A15 with $m = n$, it follows that $\hat{\Omega}_m^{2*} = \hat{\Omega}_m^2$, implying that $\hat{\Omega}^2$ is a real number. In a similar manner, through equations (15)–(18), it can be proved that, for each real $\hat{\Omega}^2$,

$$[\hat{\kappa}^2 - (\hat{\kappa}^2)^*]\int_0^h YY^*\,\mathrm{d}y = 0, \quad \text{(A16)}$$

and hence $\hat{\kappa}^2$ is a real number. Finally, according to equation (30), $\hat{\lambda}^2$ is also a real number.

For two different frequencies ($m \neq n$), there exists the following orthogonality relation of the natural vibration:

$$\frac{1}{c^2}\int_\Gamma P_n P_m^* \,\mathrm{d}\Gamma + EJ\rho_f\left[\int_0^h U_{1n}'' U_{1m}^{*\prime\prime}\,\mathrm{d}y + \int_h^H U_{2n}'' U_{2m}^{*\prime\prime}\,\mathrm{d}y\right]$$

$$+\frac{\beta}{g}\int_0^\infty P_n(x,h)P_m^*(x,h)\,\mathrm{d}x = 0, \qquad m \neq n. \quad \text{(A17)}$$

APPENDIX B: A DISCUSSION OF THE SOMMERFELD CONDITION

To solve wave radiation problems in an infinite domain, Sommerfeld [1] proposed a radiation condition at infinity. Physically, this represents a disturbance in the water propagating in the positive x direction with no wave reflected. Thus for a pressure wave in the form of a harmonic function of time $p = P(x, y)\,\mathrm{e}^{-\mathrm{i}\hat{\Omega}t}$ associated with the parameter $\hat{\Omega}$, the corresponding radiation condition is expressible as

$$p(x,y,t) = P(x,y)\,\mathrm{e}^{-\mathrm{i}\hat{\Omega}t}, \quad P_{,x} - \mathrm{i}\hat{\lambda}P = 0, \qquad x \to \infty, \quad \text{(B1)}$$

where $\hat{\Omega}$ and $\hat{\lambda}$ are of necessity two non-negative real parameters associated with the wave and $P_{,x} = \partial P/\partial x$. Because the wave equation (1) is a real valued equation, the complex conjugate (represented by a superscript asterisk) form $p^*(x, y, t)$ of the complex pressure $p(x, y, t)$ must also be a solution of equation (1). Thus the radiation condition corresponding to this conjugate solution $p^*(x, y, t)$ satisfies the expressions

$$p^*(x,y,t) = P^*(x,y)\,\mathrm{e}^{\mathrm{i}\hat{\Omega}t}, \quad P_{,x}^* + \mathrm{i}\hat{\lambda}P^* = 0, \qquad x \to \infty. \quad \text{(B2)}$$

It follows that, for the function $p(x, y, t) = P(x, y)\,\mathrm{e}^{-\mathrm{i}\hat{\Omega}t} = X(x)Y(y)T(t)$ satisfying the radiation condition (B1) at infinity, the function $X(x)$ satisfies the equation

$$(X' - \mathrm{i}\hat{\lambda}X) = 0, \quad x \to \infty, \qquad \text{radiation case.} \quad \text{(B3)}$$

and from equations (31), (32), (34), (35) and (B3), the functions of $T(t)$ and $X(x)$ satisfying equation (B1) (constant coefficients neglected) are of the forms

$$X(x)T(t) = 1, \quad \hat{\Omega} = 0 = \hat{\lambda}, \qquad X(x)T(t) = e^{i(\hat{\lambda}x - \hat{\Omega}t)}, \quad \hat{\Omega} > 0, \hat{\lambda} \geqslant 0. \qquad \text{(B4, B5)}$$

B.1. SOLUTIONS FOR THE FUNCTIONS $X(x)$, $Y(y)$ AND $T(t)$

In the case in which free surface waves are neglected, the solutions for the function $Y(y)$ expressed in equations (42) and (43) remain valid. For the case $\hat{\Omega} = 0 = \hat{\lambda}$, it follows from equation (30) that $\hat{\kappa} \equiv 0$ and, furthermore, that through equation (42) there exists no non-trivial solution. For the case $\hat{\Omega} > 0$ and $\hat{\lambda} \geqslant 0$ in combination with equations (30), (43), (B4) and (B5), the non-trivial solutions for the functions $Y(y)$, $X(x)$ and $T(t)$ satisfying equation (B1), are

$$Y_n(y) = \cos(\hat{\kappa}_n y), \qquad X_n(x)T(t) = e^{i(\hat{\lambda}_n x - \hat{\Omega}t)}, \tag{B6}$$

$$\hat{\kappa}_n = (2n-1)\pi/2h, \quad \hat{\lambda}_n^2 = (\hat{\Omega}^2/c^2) - \hat{\kappa}_n^2 \geqslant 0, \qquad \hat{\Omega} > 0, \quad n = 1, 2, 3, \ldots .$$

When free surface waves are included, the solutions for the function $Y(y)$ expressed in equations (45)–(52) and the solutions for the function $X(x)T(t)$ presented in equations (B4) and (B5) are valid. For the case $\hat{\Omega} = 0 = \hat{\lambda}$, it follows from equation (30) that $\hat{\kappa} \equiv 0$ and the only possible solutions for $T(t)$, $X(x)$ and $Y(y)$ are the constant pressure solutions

$$Y(y) = d, \quad X(x)T(t) = 1, \qquad \hat{\Omega} = 0 = \hat{\lambda}, \quad \hat{\kappa} = 0. \tag{B7}$$

For the case $\hat{\Omega} > 0$ and $\hat{\lambda} \geqslant 0$, the non-trivial solutions for the functions of $Y(y)$, $X(x)$ and $T(t)$, which satisfy equation (B1), take the forms

$$Y_n(y) = \cos(\hat{\kappa}_n y), \qquad X_n(x)T(t) = e^{i(\hat{\lambda}_n x - \hat{\Omega}t)}, \tag{B8}$$

$$\tan(\hat{\kappa}_n h) = -\hat{\Omega}^2/g\hat{\kappa}_n, \quad \hat{\lambda}_n^2 = (\hat{\Omega}^2/c^2) - \hat{\kappa}_n^2 \geqslant 0, \qquad n = 0, 1, 2, 3, \ldots,$$

$$\hat{\kappa}_n h \in ((2n-1)\pi/2, n\pi) \quad \text{for } n > 0, \qquad \hat{\kappa}_0 = i\kappa_0, \qquad \hat{\Omega} > 0.$$

B.2. CONSTANT PRESSURE SOLUTION, $\hat{\Omega}^2 = 0$

In this case the only possible solutions for the functions $X(x)$, $Y(y)$ and $T(t)$ are given in equations (44), (53) and (B7). The equation on the interaction interface expressed in equation (29) reduces to $X'(0)Y(y) \equiv 0$, for which only $\lambda_n = 0$ in equation (B7) provides a non-trivial solution. This represents a static constant fluid pressure solution and it appears only in the case of free surface waves and the radiation boundary at infinity. For this constant pressure solution, the corresponding boundary conditions are described by equations (3) and (B1). For illustrative purposes, let $F = 1$ and $d = 24EJ\hat{p}$ in equation (B7). The substitution of this solution in equation (B7) and $\hat{\Omega}^2 = 0$ into equations (21)–(26) gives

$$U_1(y) = -\hat{p}y^4 + D_1 y^3 + D_2 y^2, \qquad U_2(y) = D_3(\xi - 1) + D_4. \qquad \text{(B9, B10)}$$

Further from equations (27) and (28), it follows that

$$D_1 = 4\hat{p}h, \qquad D_2 = -6\hat{p}h^2, \qquad D_3 = -4\hat{p}h^3, \qquad D_4 = \hat{p}h^3(h - 4H). \tag{B11}$$

This result describes a zero frequency mode of the beam with constant fluid pressure $\hat{p}$: i.e., a static solution.

B.3. EIGENVALUE EQUATIONS

The possible solutions for the functions $T(t)$, $Y(y)$ and $X(x)$ are given by equations (B6) for the case in which free surface waves are neglected and by equations (B8) for the case in which waves are present on the free surface. For these two situations, the complex pressure amplitude $X(x)Y(y)$ takes the form

$$X(x)Y(y) = \sum_{n=0}^{n_1-1} G_n \cos(\hat{\kappa}_n y)\, e^{i\hat{\lambda}_n x}. \tag{B12}$$

Here n_1 denotes the same positive integer as used in equation (54) (i.e., a minimum positive integer satisfying the condition $\hat{\kappa}_n^2 > \hat{\Omega}^2/c^2$) and G_n represents a series of unknown complex constants. Because $n = 0$ does not represent a valid solution of equations (B6), $G_0 \equiv 0$ defines the case in which free surface waves are neglected. By using the approach described in section 5 and the complex beam functions

$$\begin{aligned} &\phi_1(\xi) = e^{i\omega\xi}, \quad \phi_2(\xi) = e^{-i\omega\xi}, \qquad \phi_3(\xi) = e^{\omega\xi}, \quad \phi_4(\xi) = e^{-\omega\xi}, \\ &\phi_5(\xi) = e^{i\omega(\xi-1)}, \quad \phi_6(\xi) = e^{-i\omega(\xi-1)}, \qquad \phi_7(\xi) = e^{\omega(\xi-1)}, \quad \phi_8(\xi) = e^{-\omega(\xi-1)}, \end{aligned} \tag{B13}$$

it can be shown that the equivalent equations obtained are in the forms expressed in equations (55)–(68) but with the replacement of $-\bar{\lambda}_n$ by $i\bar{\lambda}_n$ and $\sum_{n=n_1}^{\infty}$ by $\sum_{n=0}^{n_1-1}$. For example, here E_n and $\tilde{E}_n$ take the new forms

$$E_n = 1 \Bigg/ \left\{ I_n \left[\frac{1}{\bar{\kappa}_n^4 - \omega^4} + \frac{i\bar{\lambda}_n}{\gamma\omega^4} \right] \right\}, \qquad \tilde{E}_n = -1 \Bigg/ \left\{ I_n \left[\frac{1}{\bar{\kappa}_n^4 - \omega^4} + \frac{i\bar{\lambda}_n(\bar{\kappa}_n^4 - \omega^4)}{\gamma\omega^4} \right] \right\}. \tag{B14}$$

For this case, all the unknown constants in equations (56)–(60), (62)–(64) and (66)–(67), equation (65) and the characteristic eigenvalue equation (68) are complex.

B.4. DISCUSSION

Equation (A15) is valid for the radiation conditions (B1) and (B2). From this equation it follows that there exists a real frequency $\hat{\Omega}_m^2$ of the eigenvalue equation (68) if and only if $J_{mm}(\infty) = 0$. From equation (B1) and its conjugate equation (B2) it follows from equation (A12) that

$$J_{mn}(\infty) = i(\hat{\lambda}_n + \hat{\lambda}_m^*) \lim_{x\to\infty} \int_0^h P_m^*(x, y) P_n(x, y)\, dy. \tag{B15}$$

From equations (A12) and (B15) the possible cases of $J_{mm}(\infty) = 0$ are obtained as follows:
(a) $\hat{\lambda}_m = 0$—this is the solution corresponding to the constant pressure solution with $\hat{\Omega}_m = 0$ expressed in equations (B7), and since $P = 1$ and $P_{,x} = 0$, then the equation (A12) $J_{mm}(x) = 0$;
(b) $\hat{\lambda}_m = i\lambda_m$ $(\lambda_m > 0)$—that is,

$$\hat{\lambda}^m + \hat{\lambda}_m^* = 0, \tag{B16}$$

giving the solution for the undisturbed condition at infinity and therefore for a finite x the integral $J_{mm}(x) \neq 0$ but $J_{mn}(\infty) = 0$; and the undisturbed condition at infinity is obtained here because of the relaxation of the requirement for a real valued $\hat{\lambda}_m$ in the Sommerfeld condition;
(c) $\hat{\lambda}_m > 0$—this requires that

$$\lim_{x\to\infty} \int_0^h P_m^* P_m\, dy = 0, \qquad \lim_{x\to\infty} |P_m| = 0; \tag{B17}$$

(d) real function $P_m = P_m^*$ gives $J_{mm}(x) = 0$.

However, the requirements in (c) and (d) are not valid for the solutions in equations (B6) and (B8). Thus there exists no real solution of $\hat{\Omega}_m^2 \neq 0$ for the eigenvalue equation (68) under the Sommerfeld condition (B1). A further discussion of and calculations for the case of the Sommerfeld condition are not pursued in the present paper.